# Absolutely singular dynamical foliations

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## Introduction

Let $A_2$ be the automorphism of the 2-torus, $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, given by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Let $A_3$ be the automorphism of the 3-torus $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$ given by $\begin{pmatrix} A_2 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $\mathrm{Diff}^2_\mu(\mathbf{T}^3)$ be the set of $C^2$ diffeomorphisms of $\mathbf{T}^3$ that preserve Lebesgue-Haar measure $\mu$.

In [SW1], M. Shub and A. Wilkinson prove the following theorem.

**Theorem:** *Arbitrarily close to $A_3$ there is a $C^1$-open set $U \subset \mathit{Diff}^2_\mu(\mathbf{T}^3)$ such that for each $g \in U$,*

1. *$g$ is ergodic.*
2. *There is an equivariant fibration $\pi : \mathbf{T}^3 \to \mathbf{T}^2$ such that $\pi g = A_2 \pi$ The fibers of $\pi$ are the leaves of a foliation $\mathcal{W}^c_g$ of $\mathbf{T}^3$ by $C^2$ circles. In particular, the set of periodic leaves is dense in $\mathbf{T}^3$.*
3. *There exists $\lambda^c > 0$ such that, for $\mu$-almost every $w \in \mathbf{T}^3$, if $v \in T_w\mathbf{T}^3$ is tangent to the leaf of $\mathcal{W}^c_g$ containing $w$, then*
$$\lim_{n\to\infty} \frac{1}{n} \log \|T_w g^n v\| = \lambda^c.$$
4. *Consequently, there exists a set $S \subseteq \mathbf{T}^3$ of full $\mu$-measure that meets every leaf of $\mathcal{W}^c_g$ in a set of leaf-measure 0. The foliation $\mathcal{W}^c_g$ is not absolutely continuous.*

Additionally, it is shown that the diffeomorphisms in $U$ are nonuniformly hyperbolic and Bernoullian. In this note, we prove:

**Theorem I:** *Let $g$ satisfy conclusions 1.–3. of the previous theorem. Then there exist $S \subseteq \mathbf{T}^3$ of full $\mu$-measure and $k \in \mathbf{N}$ such that $S$ meets every leaf of $\mathcal{W}_g^c$ in exactly $k$ points. The foliation $\mathcal{W}_g^c$ is* absolutely singular.

**Remark:** In A. Katok's example of an absolutely singular foliation in [Mi], the leaves of the foliation meet the set of full measure in one point. In the [SW1] examples, the set $S$ may necessarily meet leaves of $\mathcal{W}_g^c$ in more than one point, as the following argument of Katok's shows.

It follows from Theorem II in [SW2] that for $k \in \mathbf{Z}_+$ and for small $a, b > 0$, the map $g = j_{a,k} \circ h_b$ satisfies the hypotheses of Theorem I, where

$$h_b(x, y, z) = (2x + y, x + y, x + y + z + b \sin 2\pi y), \qquad \text{and}$$

$$j_{a,k}(x, y, z) = (x, y, z) + a \cos(2\pi kz) \cdot (1 + \sqrt{5}, 2, 0).$$

For $k \in \mathbf{N}$, let $\rho_k$ be the vertical translation that sends $(x, y, z)$ to $(x, y, z+\frac{1}{k})$. Note that $h_b \circ \rho_k = \rho_k \circ h_b$ and $j_{a,k} \circ \rho_k = \rho_k \circ j_{a,k}$. Thus $g \circ \rho_k = \rho_k \circ g$.

The fibration $\pi : \mathbf{T}^3 \to \mathbf{T}^2$ was obtained in [SW1] by using the persistence of normally hyperbolic submanifolds under perturbations. In the present case the symmetries $\rho_k$ preserve the fibers of the trivial fibration $P : \mathbf{T}^3 \to \mathbf{T}^2$ from which one starts, and also the maps $g$. Therefore the fibers of $\pi : \mathbf{T}^3 \to \mathbf{T}^2$ (i.e., the leaves of center foliation $\mathcal{W}_g^c$) are invariant under the action of the finite group $< \rho_k >$.

Let $S$ be the (full measure) set of points in $\mathbf{T}^3$ for which the center direction is a positive Lyapunov direction (i.e. for which conclusion 3 holds). Since $\rho_k(\mathcal{W}_g^c) = \mathcal{W}_g^c$, it follows that $\rho_k S = S$. If $p \in S \cap \mathcal{W}^c(p)$, then $\rho_k(p) \in \rho_k(S) \cap \rho_k(\mathcal{W}^c(p)) = S \cap \mathcal{W}^c(p)$; that is, $S \cap \mathcal{W}^c(p)$ contains at least $k$ points.

Thus Theorem I is "sharp" in the sense that we cannot say more about the value of $k$ in general. We see no reason why $k = 1$ should hold even for a residual set in $U$.

Theorem I has an interesting interpretation. Recall that a *$G$-extension* of a dynamical system $f : X \to X$ is a map $f_\varphi : X \times G \to X \times G$, where $G$ is a compact group, of the form $(x, y) \mapsto (g(x), \varphi(x)y)$. If $f$ preserves

$\nu$, and $\varphi : X \to G$ is measurable, then $f_\varphi$ preserves the product of $\nu$ with Lebesgue-Haar measure on $G$. A $\mathbf{Z}/k\mathbf{Z}$-extension is also called a $k$-point extension.

Let $\lambda$ be an invariant probability measure for a $k$-point extension of $f : X \to X$, and $\{\lambda_x\}$ the family of conditional measures associated with the partition $\{\{x\} \times G\}$. We remark that if $\lambda$ is ergodic, then each atom of $\lambda_x$ must have the same weight $1/k$ (up to a set of $\lambda$-measure 0).

Now take $g \in U$. Choose a coherent orientation on the leaves of $\{\pi^{-1}(x)\}_{x \in T^2}$. Take $h : \mathbf{T}^3 \to \mathbf{T}^2 \times \mathbf{T}$ to be any continuous change of coordinates such that $h$ restricted to $\pi^{-1}(x)$ is smooth and orientation preserving to $\{x\} \times \mathbf{T}$. We may then write $F = h \circ g \circ h^{-1} : \mathbf{T}^2 \times \mathbf{T} \to \mathbf{T}^2 \times \mathbf{T}$ in the form

$$F(x, p) = (A_2 x, \varphi_x(p))$$

where $\varphi_x : \mathbf{T} \to \mathbf{T}$ is smooth and orientation preserving. If $P : \mathbf{T}^2 \times \mathbf{T} \to \mathbf{T}^2$ is the projection on the first factor of the product, we have $P \circ h = \pi$. Therefore, writing $\lambda = h^* \mu$, we have $P^* \lambda = \pi^* \mu$. Let $\{\lambda_x\}$ be the disintegration of the measure $\lambda$ along the fibers $\{x\} \times \mathbf{T}$. By a further measurable change of coordinates, smooth along each $\{x\} \times \mathbf{T}$ fiber, we may assume that $\lambda$-almost everywhere, the atoms of $\lambda_x$ are at $l/k$, for $l = 0, \ldots, k-1$. But then $\varphi_x$ permutes the atoms cyclically, and we obtain the following corollary.

**Corollary:** *For every $g \in U$ there exists $k \in \mathbf{N}$ such that $(\mathbf{T}^3, \mu, g)$ is isomorphic to an (ergodic) $k$-point extension of $(\mathbf{T}^2, \pi^* \mu, A_2)$.*

M. Shub has observed that if $g = j_{a,k} \circ h_b$, then $\pi^* \mu$ is actually Lebesgue measure on $\mathbf{T}^2$.

# 1 Proof of Theorem I

The proof of Theorem I follows from a more general result about fibered diffeomorphisms. Before stating this result, we describe the underlying setup and assumptions.

Let $X$ be a compact metric space with Borel probability measure $\nu$, and let $f : X \to X$ be invertible and ergodic with respect to $\nu$. Let $M$ be a closed Riemannian manifold and $\varphi : X \to \mathrm{Diff}^{1+\alpha}(M)$ a measurable map. Consider

the skew-product transformation $F : X \times M \to X \times M$ given by

$$F(x, p) = (f(x), \varphi_x(p)).$$

Assume further that there is an $F$-invariant ergodic probability measure $\mu$ on $X \times M$ such that $\pi_* \mu = \nu$, where $\pi : X \times M \to X$ is the projection onto the first factor.

For $x \in X$, let $\varphi_x^{(0)}$ be the identity map on $M$ and for $k \in \mathbf{Z}$, define $\varphi_x^{(k)}$ by

$$\varphi_x^{(k+1)} = \varphi_{f^k(x)} \circ \varphi_x^{(k)}.$$

Since the tangent bundle to $M$ is measurably trivial, the derivative map of $\varphi$ along the $M$ direction gives a cocycle $D\varphi : X \times M \times \mathbf{Z} \to GL(n, \mathbf{R})$, where $n = \dim(M)$:

$$(x, p, k) \mapsto D_p \varphi_x^{(k)}.$$

Assume that $\log^+ \|D\varphi\|_\alpha \in L^1(X \times M, \mu)$, where $\| \cdot \|_\alpha$ is the $\alpha$-Hölder norm. Let $\lambda_1 < \lambda_2 \cdots < \lambda_l$ be the Lyapunov exponents of this cocycle; they exist for $\mu$-a.e. $(x, p)$ by Oseledec's Theorem and are constant by ergodicity. We call these the *fiberwise exponents* of $F$. Under the assumptions just described, we have the following result.

**Theorem II:** *Suppose that $\lambda_l < 0$. Then there exists a set $S \subseteq X \times M$ and an integer $k \geq 1$ such that*

- $\mu(S) = 1$
- *For every $(x, p) \in S$, we have $\#(S \cap \{x\} \times M) = k$.*

This has the immediate corollary:

**Corollary:** *Let $f \in \mathrm{Diff}^{1+\alpha}(M)$. If $\mu$ is an ergodic measure with all of its exponents negative, then it is concentrated on the orbit of a periodic sink.*

The corollary has a simple proof using regular neighborhoods. Our proof is a fibered version. Theorem I is also a corollary of Theorem II. For this, the argument is actually applied to the inverse of $g$, which has negative fiberwise exponents, rather than to $g$ itself, whose fiberwise exponents are positive. As we described in the previous remarks, there is a measurable change of

coordinates, smooth along the leaves of $\mathcal{W}_g^c$ in which $g^{-1}$ is expressed as a skew product of $\mathbf{T}^2 \times \mathbf{T}$.

**Remark:** Without the assumption that $f$ is invertible, Theorem II is false. An example is described by Y. Kifer [Ki], which we recall here. Let $f : \mathbf{T} \to \mathbf{T}$ be a $C^{1+\alpha}$ diffeomorphism with exactly two fixed points, one attracting and one repelling. Consider the following random diffeomophism of $\mathbf{T}$: with probability $p \in (0,1)$, apply $f$, and with probability $1-p$, rotate by an angle chosen randomly from the interval $[-\epsilon, \epsilon]$.

Let $X = (\{0,1\} \times \mathbf{T})^{\mathbf{N}}$. To generate a sequence of diffeomorphisms $f_0, f_1, \ldots$ according to the above rule, we first define $\varphi : X \to \mathrm{Diff}^{1+\alpha}(\mathbf{T})$ by

$$\varphi(\omega) = \begin{cases} f & \text{if } \omega(0) = (0,\theta), \\ R_\theta & \text{if } \omega(0) = (1,\theta), \end{cases}$$

where $R_\theta$ is rotation through angle $\theta$. Next, we let $\nu_\epsilon$ be the product of $p, 1-p$-measure on $\{0,1\}$ with the measure on $\mathbf{T}$ that is uniformly distributed on $[-\epsilon, \epsilon]$. Then corresponding to $\nu_\epsilon^{\mathbf{N}}$-almost every element $\omega \in X$ is the sequence $\{f_k = \varphi(\sigma^k(\omega))\}_{k=0}^\infty$, where $\sigma : X \to X$ is the one-sided shift $\sigma(\omega)(n) = \omega(n+1)$.

Put another way, the random diffeomorphism is generated by the (noninvertible) skew product $\tau : X \times \mathbf{T} \to X \times \mathbf{T}$, where $\tau(\omega, x) = (\sigma(\omega), \varphi(\omega)(x))$. An ergodic $\nu_\epsilon$-stationary measure for this random diffeomorphism is a measure $\mu_\epsilon$ on $\mathbf{T}$ such that $\mu_\epsilon \times \nu_\epsilon^{\mathbf{N}}$ is $\tau$-invariant and ergodic. Such measures always exist ([Ki], Lemma I.2.2), but, for this example, there is an ergodic stationary measure with additional special properties.

Specifically, for every $\epsilon > 0$, there exists an ergodic $\nu_\epsilon$-stationary measure $\mu_\epsilon$ on $\mathbf{T}$ such that, as $\epsilon \to 0$, $\mu_\epsilon \to \delta_{x_0}$, in the weak topology, where $\delta_{x_0}$ is Dirac measure concentrated on the sink $x_0$ for $f$. From this, it follows that, as $\epsilon \to 0$, the fiberwise Lyapunov exponent for $\mu_\epsilon$ approaches $\log |f'(x_0)| < 0$, which is the Lyapunov exponent of $\delta_{x_0}$. Thus, for $\epsilon$ sufficiently small, the fiberwise exponent for $\tau$ with respect to $\mu_\epsilon$ is negative. Nonetheless, it is easy to see that $\mu_\epsilon$ for $\epsilon > 0$ cannot be uniformly distributed on $k$ atoms; if $\mu_\epsilon$ were atomic, then $\tau$-invariance of $\mu_\epsilon \times \nu_\epsilon^{\mathbf{N}}$ would imply that, for every $x \in \mathbf{T}$,

$$\begin{aligned} \mu_\epsilon(\{x\}) &= p\mu_\epsilon(\{f^{-1}(x)\}) + (1-p)\int_{-\epsilon}^{\epsilon} \mu_\epsilon(\{R_\theta(x)\})d\theta \\ &= p\mu_\epsilon(\{f^{-1}(x)\}), \end{aligned}$$

which is impossible if $\mu_\epsilon$ has finitely many atoms. In fact, $\mu_\epsilon$ can be shown to be absolutely continuous with respect to Lebesgue measure (see [Ki], p. 173ff and the references cited therein). Hence invertibility is essential, and we indicate in the proof of Theorem II where it is used.

**Proof of Theorem II:** We first establish the existence of fiberwise "stable manifolds" for the skew product $F$. A general theory of stable manifolds for random dynamical systems is worked out in ([Ki], Theorem V.1.6; see also [BL]); since we are assuming that all of the fiberwise exponents for $F$ are negative, we are faced with the simpler task of constructing fiberwise regular neighborhoods for $F$ (see the Appendix by Katok and Mendoza in [KH]). We outline a proof, following closely [KH].

**Theorem 1.1** *(Existence of Regular Neighborhoods) There exists a set* $\Lambda_0 \subseteq X \times M$ *of full measure such that for* $\epsilon > 0$*:*

- *There exists a measurable function* $r : \Lambda_0 \to (0, 1]$ *and a collection of embeddings* $\Psi_{(x,p)} : B(0, q(x, p)) \to M$ *such that* $\Psi_{(x,p)}(0) = p$ *and* $exp(-\epsilon) < r(F(x, p))/r(x, p) < exp(\epsilon)$.

- *If* $\varphi_{(x,p)} = \Psi^{-1}_{F(x,p)} \circ \varphi_x \circ \Psi_{(x,p)} : B(0, r(x, p)) \to \mathbf{R}^n$*, then* $D_0\varphi_{(x,p)}$ *satisfies*

$$exp(\lambda_1 - \epsilon) \leq \|D_0\varphi^{-1}_{(x,p)}\|^{-1}, \|D_0\varphi_{(x,p)}\| \leq exp(\lambda_l + \epsilon).$$

- *The* $C^1$ *distance* $d_{C^1}(\varphi_{(x,p)}, D_0\varphi_{(x,p)}) < \epsilon$ *in* $B(0, r(x, p))$.

- *There exist a constant* $K > 0$ *and a measurable function* $A : \Lambda_0 \to \mathbf{R}$ *such that for* $y, z \in B(0, r(x, p))$,

$$K^{-1}d(\Psi_{(x,p)}(y), \Psi_{(x,p)}(z)) \leq \|y - z\| \leq A(x)d(\Psi_{(x,p)}(y), \Psi_{(x,p)}(z)),$$

*with* $exp(-\epsilon) < A(F(x, p))/A(x, p) < exp(\epsilon)$.

**Proof:** See the proof of Theorem S.3.1 in [KH]. □

Decompose $\mu$ into a system of fiberwise measures $d\mu(x, p) = d\mu_x(p)d\nu(x)$. Invariance of $\mu$ with respect to $F$ implies that, for $\nu$-a.e. $x \in X$,

$$\varphi_{x*}\mu_x = \mu_{f(x)}.$$

**Corollary 1.2** *There exists a set $\Lambda \subseteq X \times M$, and real numbers $R > 0$, $C > 0$, and $c < 1$ such that*

*(1) $\mu(\Lambda) > .5$, and, if $(x,p) \in \Lambda$, then $\mu_x(\Lambda_x) > .5$, where $\Lambda_x = \{p \in M \,|\, (x,p) \in \Lambda\}$,*

*(2) If $(x,p) \in \Lambda$ and $d_M(p,q) \leq R$, then*

$$d_M(\varphi_x^{(m)}(p), \varphi_x^{(m)}(q)) \leq Cc^m d_M(p,q),$$

*for all $m \geq 0$.*

**Proof:** This follows in a standard way from the Mean Value Theorem and Lusin's Theorem.□

To prove Theorem II, it suffices to show that there is a positive $\nu$-measure set $B \subseteq X$, such that for $x \in B$, the measure $\mu_x$ has an atom, as the following argument shows. For $x \in X$, let $d(x) = \sup_{p\in M} \mu_x(p)$. Clearly $d$ is measurable, $f$-invariant, and positive on $B$. Ergodicity of $f$ implies that $d(x) = d > 0$ is positive and constant for almost all $x \in X$. Let $S = \{(x,p) \in X \times M \,|\, \mu_x(p) \geq d\}$. Observe that $S$ is $F$-invariant, has measure at least $d$, and hence has measure 1. The conclusions of Theorem II follow immediately.

Let $\Lambda$, $R > 0$, $C > 0$, and $c < 1$ be given by Corollary 1.2, and let $B = \pi(\Lambda)$. Let $N$ be the number of $R/10$-balls needed to cover $M$. We now show that for $\nu$-almost every $x \in B$, the measure $\mu_x$ has at least one atom.

For $x \in X$, let

$$m(x) = \inf \sum \text{diam } (U_j),$$

where the infimum is taken over all collections of closed balls $U_1, \ldots, U_k$ in $M$ such that $k \leq N$ and $\mu_x(\bigcup_{j=1}^k U_j) \geq .5$. Let $m = \text{ess sup }_{x\in B} m(x)$.

We now show that $m = 0$. If $m > 0$, then there exists an integer $J$ such that

$$C\Delta c^J N \quad < \quad m/2, \tag{1}$$

where $\Delta$ is the diameter of $M$. Let $\mathcal{U}$ be a cover of $M$ by $N$ closed balls of radius $R/10$. For $x \in B$, let $U_1(x), \ldots, U_{k(x)}(x)$ be those balls in $\mathcal{U}$

that meet $\Lambda_x$. Since these balls cover $\Lambda_x$, and $\mu_x(\Lambda_x) > .5$, it follows that $\mu_x(\bigcup_{j=1}^{k(x)} U_j(x)) \geq .5$. But $\varphi_x^{(i)}{}_*\mu_x = \mu_{f^i(x)}$, and so it's also true that

$$\mu_{f^i(x)}(\bigcup_{j=1}^{k(x)} \varphi_x^{(i)}(U_j(x))) \quad \geq \quad .5, \tag{2}$$

for all $i$.

We now use the fact that $\varphi_x^{(i)}$ contracts regular neighborhoods to derive a contradiction. The balls $U_j(x)$ meet $\Lambda_x$ and have diameter less than $R/10$, and so by Corollary 1.2, (2), we have

$$\text{diam } (\varphi_x^{(i)}(U_j(x))) \quad \leq \quad C\Delta c^i. \tag{3}$$

Let $\tau : B \to \mathbf{N}$ be the first-return time of $f^J$ to $B$, so that $f^{J\tau(x)}(x) \in B$, and $f^{Ji}(x) \notin B$, for $i \in \{1, \ldots, \tau(x) - 1\}$. Decompose the set $B$ according to these first return times:

$$B = \bigcup_{i=1}^{\infty} B_i \pmod 0,$$

where $B_i = \tau^{-1}(i)$. *Because $f$ is invertible and $f^{-1}$ preserves measure*, we also have the mod 0 equivalence:

$$B' := \bigcup_{i=1}^{\infty} f^{Ji}(B_i) = B \pmod 0.$$

Let $y \in B'$. Then $y = f^{Ji}(x)$, where $x \in B_i \subseteq B$, for some $i \geq 1$. It follows from the definition of $m(y)$ and inequalities (2), (3) and (1) that

$$\begin{aligned} m(y) &\leq \sum_{j=1}^{k(x)} \text{diam } (\varphi_x^{(Ji)}(U_j(x))) \\ &\leq Ck(x)\Delta c^{Ji} \\ &\leq CN\Delta c^J \\ &< m/2. \end{aligned}$$

But then

$$\begin{aligned} m &= \text{ess sup }_{x\in B} m(x) \\ &= \text{ess sup }_{y\in B'} m(y) \\ &< m/2, \end{aligned}$$

contradicting the assumption $m > 0$.

Thus $m = 0$, and, for $\nu$-almost every $x \in B$, we have $m(x) = 0$. If $m(x) = 0$, then there is a sequence of closed balls $U^1(x), U^2(x), \cdots$ with $\lim_{i\to\infty} \text{diam } (U^i(x)) = 0$ and $\mu_x(U^i(x)) \geq .5/N$, for all $i$. Take $p_i \in U^i(x)$; any accumulation point of $\{p_i\}$ is an atom for $\mu_x$. Since we have shown that $\mu_x$ has an atom, for $\nu$-a.e. $x \in B$, the proof of Theorem II is complete.$\Box$

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